0.1 MineCraft Addition Monster

This is one of many ways to present the Euclidean algorithm. It is nice to be able to show the same mathematics in several different ways. Often students will work on a second activity and only understand near the very end that they have seen the same thing before. Of the various activities introducing the Euclidean algorithm, this is one of the easiest. The problem is framed by a story related to the popular (in 2015) computer game *MineCraft*. The main ingredient to the story is a monster chasing prey. Monsters have captured children, or at least their attention, forever so it will always be possible to frame a question via a popular monster.

Grade Level/Prerequisites: This activity is suitable for students in grades 3 – 12. Students must be able to add to do this activity. To get the most out of this activity, students should be familiar with multiplication, long division and reducing fractions. It is probably best for students in grades 4 – 6.

Time: Participants can get started in 30 minutes. It will fit fairly well in a 90 minute session. It will probably help to describe the punch line, i.e. the Euclidean algorithm either at the end of the session, or even better, then next time the group of students meet.

Materials: The only thing required is something to write with and something to write on. Paper and pencils will work. Chalk board with colored chalk, or white board with colored pens, or similar to display computations to participants. Making things large, is always fun. Sidewalk chalk will allow participants to "walk like monsters."

Preparation: Read participant hand-out and try problems. Read leader guide. Make copies of participant hand-out. Perhaps make fraction-labeled grids with sidewalk chalk.

Objectives: Let participants discover the Euclidean algorithm, and perhaps the Stern-Brocot tree.

References/Authorship: The algorithm appears in Euclid's *Elements* [**?**]. Thus many people have described it over the years. The Stern-Brocot tree was discovered by Stern and Brocot in the 1800s. Dave Auckly wrote this version of the activity based on presentations of B. Thornton and G. Pinter and the MineCraft video game.

MineCraft Addition Monster

MineCraft will introduce a new monster this year – the selective addition monster. This type of monster starts the block in the northwest corner of the map (it is called the 1/1 block.) The monster then adds the left number to the right or the right to the left and moves to the corresponding block. For example, the monster might move $1/1 \mapsto 2/1 \mapsto 3/1 \mapsto 3/4 \mapsto 3/7$. Whenever the monster lands on a block, it kills all players on, above, or below the block. These monsters can move very fast!

(1) If the monster was on block 3/5 where could the monster move?
(2) If the monster moves to block 3/5 where could it have been just before the move?
(3) Can the monster get to 1/2? to 1/3, to 1/4?
(4) Can the monster get to 2/1? to 3/1?
(5) Mark blocks that the monster can reach after some number of steps when it starts on 1/1 with some color or marker.
(6) Can the addition monster get to 15/22? If so, how?
(7) Can the addition monster get to 121/56 If so, how?
(8) If the monster started on block 2/4 by a glitch, could it get to block 301/444? Why or why not?
(9) If the monster started on block 3/9 by a glitch, what can you say about the blocks that the monster could reach?
(10) Are there any safe blocks when the monster starts on 1/1? If so which blocks are safe? Why?
(11) Draw the patten of dangerous and safe blocks.
(12) What does the previous problem have to do with fractions?
(13) Trees are planted at the centers of each square on the grid. You are standing on the square that would be labeled 0/0! Which trees can you see?
(14) Are there blocks that the monster can reach in two different ways?
(15) Are there blocks that the monster can reach in three different ways?
(16) Label each block the monster can reach with the smallest number of steps the monster can use to reach the block.

1/1	1/2	1/3	1/4	1/5	1/6	1/7	1/8	1/9	1/10	1/11	1/12	1/13	1/14	1/15	1/16	1/17	1/18	1/19	1/20	1/21	1/22	1/23	1/24	1/25	1/26	1/27	1/28
2/1	2/2	2/3	2/4	2/5	2/6	2/7	2/8	2/9	2/10	2/11	2/12	2/13	2/14	2/15	2/16	2/17	2/18	2/19	2/20	2/21	2/22	2/23	2/24	2/25	2/26	2/27	2/28
3/1	3/2	3/3	3/4	3/5	3/6	3/7	3/8	3/9	3/10	3/11	3/12	3/13	3/14	3/15	3/16	3/17	3/18	3/19	3/20	3/21	3/22	3/23	3/24	3/25	3/26	3/27	3/28
4/1																											
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6/1																											
7/1																											
8/1																											
9/1																											
10/1																											
11/1																											

Teacher Guide/Solutions

Solutions:

(1) Following the directions the monster can only add the left number to the right or the right to the left, so $3/8$ or $5/8$.

(2) To get to $3/5$ one of the numbers in the starting fraction had to be 3 and the sum of the numbers in the starting fraction had to be 5, so the starting fraction could only be $2/3$ or $3/2$.

(3) The monster can get to all three: $1/1 \mapsto 1/(1+1) = 1/2 \mapsto 1/(1+2) = 1/3 \mapsto 1/(1+3) = 1/4$. In fact all fractions of the form $1/n$ are possible.

(4) Both are possible: $1/1 \mapsto (1+1)/1 = 2/1 \mapsto (1+2)/1 = 3/1$. All fractions of the form $n/1$ are possible.

(5) Experimentally it appears that the monster can get to all of the red dots and no others.

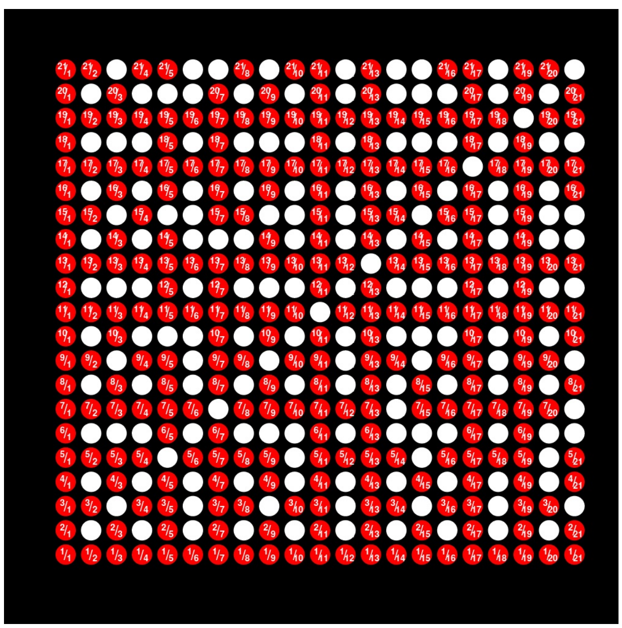

Figure 1: Danger spots

(6) The secret to solving problems 6 and 7 is to work backwards –

The monster could get to $15/22$ if it could get to $15/7$
The monster could get to $15/7$ if it could get to $8/7$

The monster could get to $8/7$ if it could get to $1/7$
The monster could get to $1/7$ if it could get to $1/6$...

In problem (3) we saw that the monster could get to $1/6$ thus the monster can get to $15/22$. To get there just follow the above steps in reverse.

(7) We will use the same steps to analyze $121/56$. This time we can take several steps at a time. This amounts to dividing and taking the remainder. Thus we will not pass from $121/56$ to $65/56$ and then to $9/56$. We will go directly to $9/56$ by noticing that 56 goes into 121 2 times (so we will combine two backward steps) with a remainder of 9 to get to $9/56$.

The monster could get to $121/56$ if it could get to $9/56$
The monster could get to $9/56$ if it could get to $9/2$
The monster could get to $9/2$ if it could get to $1/2$ and the monster can get to $1/2$.

(8) Starting from $2/4$ one can get to $2/6$ or to $6/4$. In general one can go from a/b to $a/(a+b)$ or to $(a+b)/b$. When both a and b are even, $a+b$ will be even. Thus if the top and bottom of a fraction are even a fraction generated by either monster move will still even numerator and denominator. Thus $301/444$ is not reachable from $2/4$.

(9) Starting from $3/9$ one can get to $3/12$ or to $12/9$. In general one can go from a/b to $a/(a+b)$ or to $(a+b)/b$. When both a and b are divisible by 3, $a+b$ will be divisible by 3. Thus if the top and bottom of a fraction have a common factor of 3 a fraction generated by either monster move will still have a common factor of 3 in the numerator and denominator. Thus only fractions with a common factor of 3 in the numerator and denominator will be reachable from $3/9$.

(10) Any non-reduced fraction can only be connected to other fractions with the same common factors in the numerator and denominator. In greater detail, starting from na/nb one can pass to $na/(na+nb) = na/n(a+b)$ or to $(na+nb)/nb = n(a+b)/nb$. Conversely, to reach na/nb one must start from $na/(na-nb) = na/n(a-b)$ or from $(na-nb)/nb = n(a-b)/nb$. Thus any non-reduced fraction is safe.

(11) The argument in problem (10) shows that the monster can get to all of the red dots and no others.

(12) We have been using the language of reduced fractions to describe the pattern. The reduced fractions are dangerous, and the un-reduced fractions are safe.

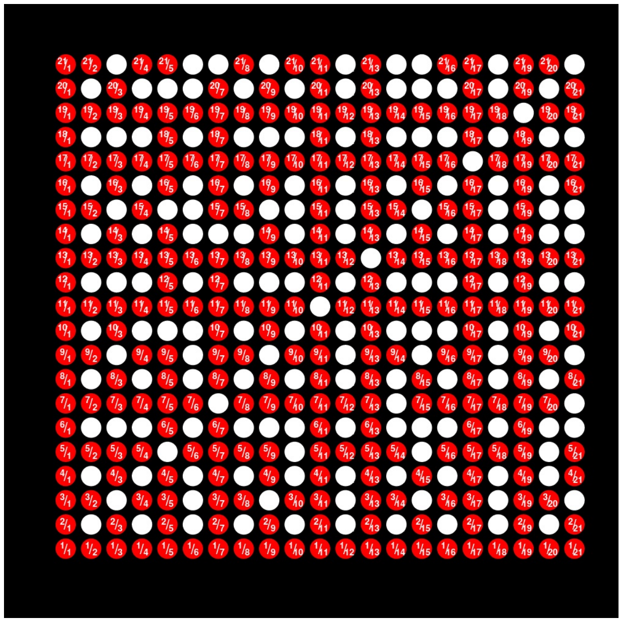

Figure 2: Danger spots

(13) From the spot labeled 0/0 one can see 1/0, 0/1 and 1/1. One cannot see 2/0, 0/2 or 2/2 because each of these is blocked, e.g. the view of 2/2 is blocked by 1/1. In fact when $n > 1$, the view of the tree at na/nb is blocked by the tree at a/b. Conversely, if the fraction of the tree is reduced, then no other tree blocks the view, so such trees are visible. The tree version of the problem is a good warm-up problem for other sessions. When the tree version is used in a different lesson it is nice for students to discover the pattern in a larger problem.

(14) One could connect each labeled dot, say a/b, to the dots related to it by monster moves, so $a/(a+b)$ and $(a+b)/b$. The resulting (directed) graph is the *Stern-Brocot* tree. A tree is a graph with no cycles. To see that it is a tree, so there is a unique way to get to each point, work backwards. The only fractions that could generate a/b by monster moves are $(a-b)/b$ or $a/(b-a)$. With the exception of 1/1 the numerator and denominator of any reduced fraction will be distinct. Thus one of the numbers $(a-b)/b$ or $a/(b-a)$ would be negative and no negative number could appear via monster moves. Thus each reduced fraction may only be reached in just one way. (The fraction 1/1 can only be the starting point.)

(15) There is only one way to reach each reduced fraction, so there are no

fractions that may be reached in three ways. (See the answer to problem (15).

(16) There is no easy answer to the general case of this question. Perhaps the best general way to figure out how many monster steps it takes to reach a number is to just work backwards. Let $N(a/b)$ denote the number of monster steps required. We will declare that starting at $1/1$ is the first step. One way to organize this is to write out the Stern-Brocot tree. Then the elements in the first row are reachable in 1 step, the elements in the second row in 2-steps, etc.

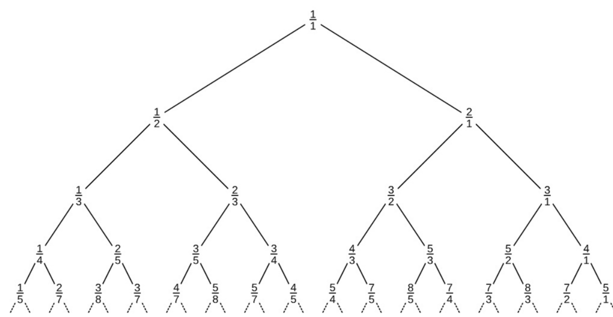

Figure 3: Stern-Brocot Tree

An alternate way to list the possibilities is to make a table as follows:

7	6	5	4	3	2	1	a/b
					1	2	2
				1	3	3	3
			1	4	2	4	4
		1	5	4	4	5	5

Some patterns that emerge are $N(a/b) = N(b/a)$ so the above table is symmetric. Common factors do not matter, so $N(na/nb) = N(a/b)$. One may view as the computational complexity of the problem. Even though it is difficult to give a formula for $N(a/b)$ it is possible to compute some special cases. The worst case occurs for fractions of the form $1/n$. One has $N(1/n) = n$ as can be seen via induction the way problem (3) was solved. The best case is for non-reduced fractions takes place for fractions of the form a/a for which one has $N(a/a) = 1$. To see the best case for reduced fractions consider the largest numbers that can be generated in a given number of steps. In 1 step we have $1/1$. In 2 steps we have $1/2$. From here we can either keep the large number or keep the small number. If we are trying to build the largest number possible we will always keep the largest two numbers. Thus it will continue $3/2$ then $3/5$, $8/5$, etc. Ignoring which number is on the numerator and which is on the denominator (reasonable because $N(a/b) = N(b/a)$, we see that the largest numbers that can be generated in N steps are the consecutive elements of the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

This is known as the *Fibonacci sequence.* It is specified by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. We have $N(F_n/F_{n+1}) = n$. It is possible to derive an explicit formula for the nth term of the Fibonacci sequence and translate it into a lower bound on the number of steps required for a/b in terms of the max of a and b when a and b have no common factor. It would also be possible to compute the average number of steps required for a/b in terms of the max of a and b. Both of these last questions are beyond the scope of the expected audience for this activity.

Additional Background: The Euclidean Algorithm can be used to solve a variety of related problems.

Greatest Common Divisor To find the greatest common divisor of two positive integers a and b subtract the small one from the large one and repeat. Eventually the process will stop at 0 and d where $d = \gcd(a, b)$.

Reduce fractions Replace a/b by $(a/\gcd(a, b))/(b/\gcd(a, b))$

Least Common Multiple Use $\operatorname{lcm}(a, b) = ab/\gcd(a, b)$.

Linear Diophantine Equations To find integers x and y so that $ax + by = c$, notice that $a(1)+b(0) = a$ and $a(0)+b(1) = b$. Repeatedly subtract the the equation with smallest constant term from the equation with second smallest constant term. This will end with the second smallest constant term equal to $\gcd(a, b)$ and the smallest constant term equal to zero. If $\gcd(a, b)$ does not divide c there can be no solution. If $\gcd(a, b)$ does divide c, multiply the second last equation by $c/\gcd(a, b)$. For example to solve $109x + 52y = 13$:

$$\begin{aligned} 109(1) + 52(0) &= 109 \\ 109(0) + 52(1) &= 52 \\ 109(1) + 52(-1) &= 57 \\ 109(1) + 52(-2) &= 5 \\ &\cdots \\ 109(-10) + 52(21) &= 2 \\ 109(21) + 52(-44) &= 1 \end{aligned}$$

Thus $109(273) + 52(-572) = 13$. Analogues of the above computations also work with polynomials and with elements of any type of collection of "numbers" of a type known as a *Euclidean Domain*.

Periodic lattices are ubiquitous in math and science. In two dimensions such a periodic lattice may be modeled by the collection of ordered pairs of integers:

$$\mathbb{Z}^2 := \{(n, m) \mid n, m \in \mathbb{Z}\} .$$

The invertible linear maps preserving this structure are just the invertible 2×2 matrices with integer coefficients and determinant equal to ± 1. This collection of matrices is denoted by $\mathrm{GL}_2(\mathbb{Z})$.

The idea behind the Euclidean algorithm may be used to show that the group $\mathrm{GL}_2(\mathbb{Z})$ is generated by the matrices:

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} , \text{ and } R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

In fact these same matrices encode the steps in the Euclidean algorithm. Indeed:

$$T^{-1} \begin{bmatrix} 109(1) + 52(0) = 109 \\ 109(0) + 52(1) = 52 \end{bmatrix} = \begin{bmatrix} 109(1) + 52(-1) = 57 \\ 109(0) + 52(1) = 52 \end{bmatrix} , \text{ and}$$

$$R \begin{bmatrix} 109(1) + 52(-1) = 57 \\ 109(0) + 52(1) = 52 \end{bmatrix} = \begin{bmatrix} 109(0) + 52(1) = 52 \\ 109(1) + 52(-1) = 57 \end{bmatrix} .$$

If one projects vectors to ratios equating (x, y) with $[x : y]$ one would have

$$T(x/y) = T([x : y]) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ x \end{bmatrix} = [x{+}y : y] = (x{+}y)/y = x/y{+}1 .$$

Thus $T(z) = z{+}1$ and $T^{-1}(z) = z{-}1$. Similarly one would see that $R(z) = 1/z$. If one wished to restrict to $2{\times}2$ matrices with determinant equal to 1, one could replace the operation R by the operation $S(z) = -1/z$ and the corresponding matrix.

Presentation:

This set of problems lead the participants more than many sets of math circle problems. Thus this activity is good for participants that have not had that much math circle experience.

Monsters are always a good way to engage an audience. Dave first used this activity around Halloween and drew grids with sidewalk chalk outside of the school. Having the students "monster walk" from the classroom to the chalk in front of the school was a playful way to start. MineCraft is based on worlds created from cubical blocks. Dave explained that the monster could jump between squares according to the rules instantly, and killed everything above and below the labeled square.

If participants are not familiar with MineCraft, the activity leader may pose the problem in terms of a better known monster concept.

It is very helpful to have grids prepared before the activity, so students don't have to spend time making the grids. Students can work in groups to solve the problems. The activity leader will first have to make sure that everyone understands the rules for how the monster moves. Once this is the case, let the students start working through the problems in the list. When a student has a solution have them explain it and explain why their solution is correct.

Don't worry if students do not get all the way through all of the problems. The last problem is sufficiently complicated, that no one will complete it in the allotted time. The goal here is to get participants engaged and interested in a mathematics question.

In an ideal situation the activity will meet with the same group of participants many times. If this is the case, it will be possible to give the tree problem (13) as a warm-up problem to a different activity. This activity may be run at a later date without rushing. Participants can continue to think about it, and the leader can go over the Euclidean algorithm at a different meeting as a way of summarizing what happened during this activity.

Dan Myers describes "3 acts" as a model for a math lesson.

Act 1 – Motivate a Question: The goal here is to show something so compelling, people will want to explore it and get some answers.

Act 2 – Work and Learn: Let participants explore the topic, discovering things that will help them answer the question.

Act 3 – Summarize: Make sure that participants see some answers and get the point. (This can take place at a later meeting.)